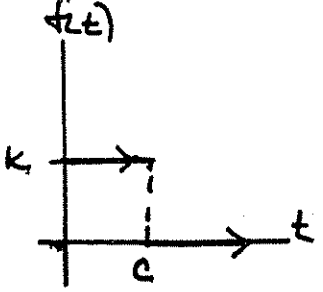
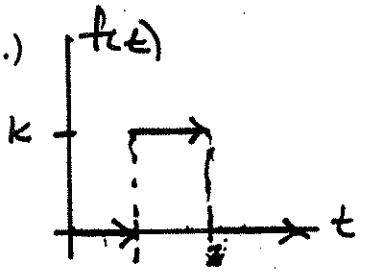
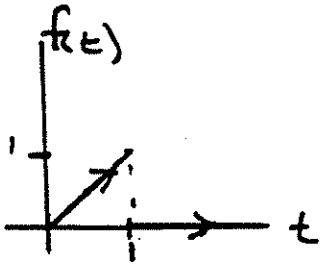
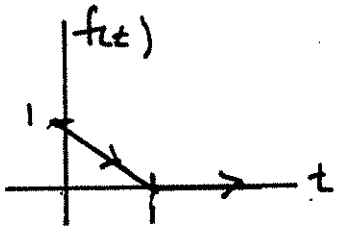
Find the Laplace Transforms

1.) 

2.) 

3.) 

4.) 

5.) $t + 2$

6.) $at + b$

7.) $a + bt + ct^2$

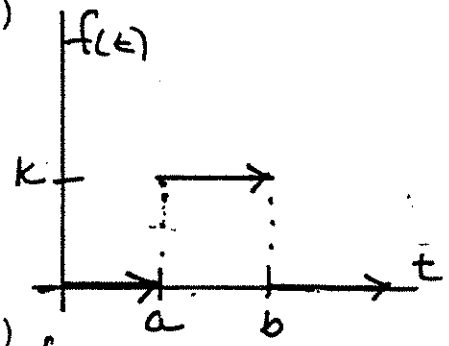
8.) $4t^3 + t^2$

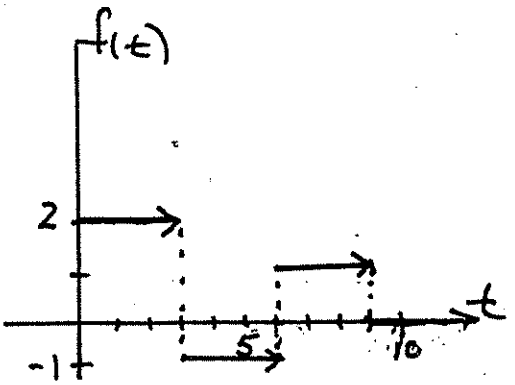
9.) $a \cos 2t$

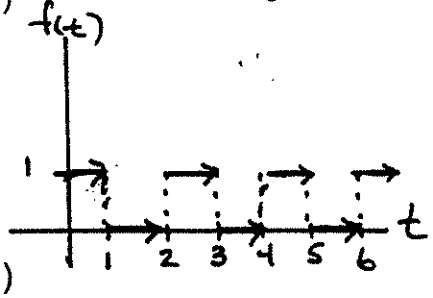
10.) $e^{-at + b}$

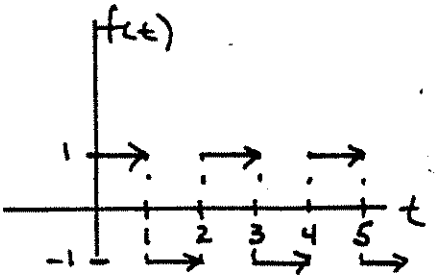
11.) $\sin(\omega t + \theta)$

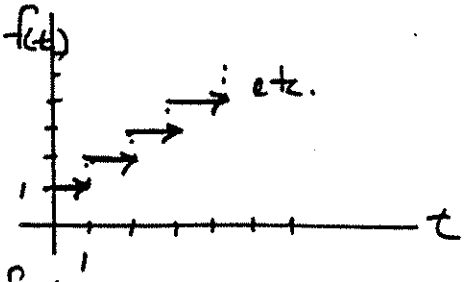
12.) $\cosh^2 3t$

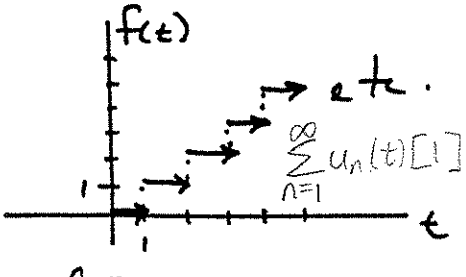
13.) 

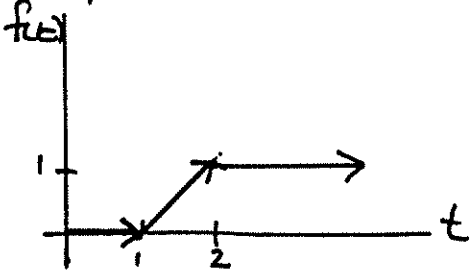
14.) 

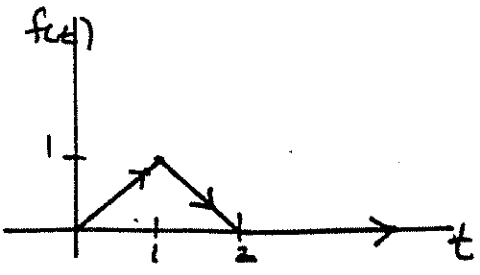
15.) 

16.) 

17.) 

18.) 

19.) 

20) 

21.) $u_1(t)[t-1]$ 22.) $u_1(t)[t]$ 23.) $u_\pi(t)[\cos t]$

24.) $u_{2\pi/\omega}(t)[\sin\omega t]$ 25.) $u_a(t)[e^{kt}]$

26.) $u_1(t)[t^2]$ 27.) te^t 28.) t^2e^{2t}

29.) $t \sin 3t$ 30.) t^2e^{-t} 31.) $t^2 \cos t$

32) $te^{-t}\cos t$ 33.) $te^{-t}\cosh 2t$ 34.) $t^2\sinh 2t$

35.) $te^{-2t}\sin t$ 36.) $1*1$ 37.) $1*\sin t$

38.) e^t*e^t 39.) $e^{at}*e^{bt}$ 40.) $t*e^{at}$

41.) $\sin t * \sin t$ 42.) $\sin t * \sin 2t$ 43.) $\cos t * \cos t$

44.) $\sin t * \cos t$

The following are assumed to be periodic with period 2π

45.) $f(t) = 2\pi - t \quad (0 < t < 2\pi)$

46.) $f(t) = e^t \quad (0 < t < 2\pi)$

47.) $f(t) = t^2 \quad (0 < t < 2\pi)$

48.) $f(t) = 1$ when $0 < t < \pi$, $f(t) = -1$ when $\pi < t < 2\pi$

49.) $f(t) = t$ when $0 < t < \pi$, $f(t) = \pi - t$ when $\pi < t < 2\pi$

50.) $f(t) = 0$ when $0 < t < \pi$, $f(t) = t - \pi$ when $\pi < t < 2\pi$

Answers to the transforms:

1.) $k(1 - e^{-cs})/s$ 2.) $k(e^{-s} - e^{-2s})/s$

3.) $-e^{-s}/s + (1 - e^{-s})/s^2$ 4.) $1/s - (1 - e^{-s})/s^2$

5.) $1/s^2 + 2/s$ 6.) $a/s^2 + b/s$

7.) $a/s + b/s^2 + 2c/s^3$ 8.) $24/s^4 + 2/s^3$

9.) $as/(s^2 + 4)$ 10.) $e^b/(s + a)$

11.) $(\omega \cos \theta + s \sin \theta)/(s^2 + \omega^2)$

12.) $\cosh^2 3t = (\cosh 6t)/2, \quad s/(2s^2 - 72) + 1/2s$

13.) $k(e^{-as} - e^{-bs})/s$ 14.) $(2 - 3e^{-3s} + 2e^{-6s} - e^{-9s})/s$

15.) $1/s(1 + e^{-s})$ 16.) $(1 - e^{-s})/s(1 + e^{-s})$

17.) $1/s(1 - e^{-s})$ 18.) $e^{-s}/s(1 - e^{-s})$

19.) $(e^{-s} - e^{-2s})/s^2$ 20.) $(1 - 2e^{-s} + e^{-2s})/s^2$

21.) e^{-s}/s^2 22.) $e^{-s}(s^{-2} + s^{-1})$

23.) $-se^{-\pi s}/(s^2 + 1)$ 24.) $\omega e^{-2\pi s/\omega}/(s^2 + \omega^2)$

25.) $e^{a(k - s)}/(s-k)$ 26.) $e^{-s}(2s^{-3} + 2s^{-2} + s^{-1})$

27.) $1/(s - 1)^2$ 28.) $2/(s-2)^3$

29.) $6s/(s^2 + 9)^2$ 30.) $2/(s + 1)^3$

31.) $2s(s^2 - 3)/(s^2 + 1)^3$ 32.) $(s^2 + 2s)/(s^2 + 2s + 2)^2$

33.) $(s^2 + 2s + 5)/(s^2 + 2s - 3)^2$ 34.) $(12s^2 + 16)/(s^2 - 4)^3$

35.) $2(s + 2)/(s^2 + 4s + 5)^2$ 36.) t

37.) $1 - \cos t$ 38.) te^t

39.) $(e^{at} - e^{bt})/(a-b)$ 40.) $(e^{at} - 1)/a^2 - t/a$

41.) $(1/2)(\sin t - t \cos t)$ 42.) $(2/3) \sin t - (1/3) \sin 2t$

43.) $(1/2)(t \cos t + \sin t)$ 44.) $(t/2) \sin t$

45.) $\dfrac{2\pi s + e^{-2\pi s} - 1}{s^2(1 - e^{-2\pi s})}$. 46.) $\dfrac{e^{2(1-s)\pi} - 1}{(1 - s)(1 - e^{-2\pi s})}$

47.) $\dfrac{2(e^{2\pi s} - 1 - 2\pi s - 2\pi^2 s^2)}{s^3(e^{2\pi s} - 1)}$ 48.) $\dfrac{(1 - e^{-\pi s})^2}{s(1 - e^{-2\pi s})}$

49.) $[\dfrac{\pi}{s} e^{-\pi s}(e^{-\pi s} - 1) + \dfrac{1}{s^2}(e^{-\pi s} - 1)^2]/(1-e^{-2\pi s})$

50.) $[s^{-2}(e^{-\pi s} - e^{-2\pi s}) - \pi s^{-1} e^{-2\pi s}]/(1 - e^{-2\pi s})$

Find the Inverse Transforms

1.) $\frac{1}{s^2 + 9}$

2.) $\frac{3}{s + \pi}$

3.) $\frac{a_1}{s} + \frac{a_2}{s^2} + \frac{a_3}{s^3}$

4.) $(2s + 1)/(s^2 + 4)$

5.) $4(s + 1)/(s^2 - 16)$

6.) $\frac{2}{s} + \frac{1}{s + 2}$

7.) $\frac{s}{s^2 + n^2\pi^2}$

8.) $\frac{1}{(s + 1)(s + 2)}$

9.) $\frac{3}{s^2 + 3s}$

10.) $\frac{1}{s(s - 2)}$

11.) $\frac{1}{s(s^2 + 9)}$

12.) $\frac{1}{s(s^2 - 1)}$

13.) $\frac{1}{s^2(s + 1)}$

14.) $\frac{1}{s^2}\left(\frac{s - 1}{s + 1}\right)$

15.) $\frac{1}{s^2}\left(\frac{s - 2}{s^2 + 4}\right)$

16.) $\frac{54}{s^3(s - 3)}$

17.) $\frac{2s - \pi}{s^3(s - \pi)}$

18.) $\frac{s + 3}{(s + 1)^2 + 1}$

19.) $\frac{2s + 1}{s^2 + 4s + 13}$

20.) $\frac{1}{(s-2)^3} + \frac{1}{(s - 2)^5}$

21.) $\frac{bs + c}{(s - a)^2 + \omega^2}$

22.) $(e^{-3s} - e^{-s})/s$

23.) e^{-s}/s^2

24.) $(e^{-s} + e^{-2s} - 3e^{-3s} + e^{-6s})/s^2$

25.) $(2e^{-as} - 1)/s$

26.) e^{-2s}/s

27.) e^{-s}/s^3

28.) $e^{-2s}/(s - 2)$

29.) $e^{-s}/(s^2 + \pi^2)$

30.) $e^{-s}/(s-3)$

31.) $(1 - e^{-\pi s})/(s^2 + 4)$

32.) $e^{-\pi s}/(s^2 + 2s + 2)$

33.) $s(1 + e^{-\pi s})/(s^2 + 1)$

34.) $\dfrac{1}{(s - 4)^2}$

35.) $2s/(s^2 + 1)^2$

36.) $2/(s - a)^3$

37.) $\operatorname{arccot}(s + 1)$

38.) $\operatorname{arc\,cot} \dfrac{s}{\omega}$

39.) $\ln \dfrac{s}{s - 1}$

40.) $\ln \dfrac{s + a}{s - a}$

41.) $\ln \dfrac{s + a}{s + b}$

42.) $\ln \dfrac{s^2 + 1}{(s - 1)^2}$

43.) $1/(s - a)^2$

44.) $1/(s - a)^3$

45.) $\dfrac{1}{(s - a)(s - b)}$

46.) $\dfrac{1}{s(s^2 + \omega^2)}$

47.) $\dfrac{1}{s^2(s^2 + \omega^2)}$

48.) $1/(s^2 + 1)^2$

49.) $s/(s^2 + \omega^2)^2$

50.) $s^2/(s^2 + 1)^2$

51.) $\dfrac{s}{(s^2 + a^2)(s^2 + b^2)}$

52.) $\dfrac{3s - 2}{s^2 - s}$

53.) $\dfrac{s^2 + 9s - 9}{s^3 - 9s}$

54.) $\dfrac{4s + 4}{s^2 + 16}$

55.) $s/(s^2 + 2s + 2)$

56.) $s/(s + 1)^2$

57.) $\dfrac{s^3 + 6s^2 + 14s}{(s + 2)^4}$

58.) $\dfrac{2s^2 - 3s}{(s - 2)(s - 1)^2}$

59.) $\dfrac{s^2 + 2s}{(s^2 + 2s + 2)^2}$

60.) $\dfrac{s^2 - 6s + 7}{(s^2 - 4s + 5)^2}$

61.) $\dfrac{3s^2 - 6s + 7}{(s^2 - 2s + 5)^2}$

62.) $\dfrac{s^3 - 3s^2 + 6s - 4}{(s^2 - 2s + 2)^2}$

Answers to inverses:

1.) $(1/3)\sin 3t$

2.) $3e^{-\pi t}$

3.) $a_1 + a_2 t + a_3 t^2/2$

4.) $2\cos 2t + 1/2 \sin 2t$

5.) $4\cosh 4t + \sinh 4t$

6.) $2 + e^{-2t}$

7.) $\cos n\pi t$

8.) $e^{-t} - e^{-2t}$

9.) $1 - e^{-3t}$

10.) $\dfrac{e^{2t}}{2} - \dfrac{1}{2}$

11.) $(1 - \cos 3t)/9$

12.) $\cosh t - 1$

13.) $t + e^{-t} - 1$

14.) $2(1 - e^{-t}) - t$

15.) $(\sin 2t - \cos 2t)/4 + 1/4 - t/2$

16.) $2e^{3t} - 9t^2 - 6t - 2$

17.) $t^2/2 - t/p + (e^{\pi t} - 1)/\pi^2$

18.) $e^{-t}(\cos t + 2 \sin t)$

19.) $e^{-2t}(2\cos 3t - \sin 3t)$

20.) $e^{2t}\left(\dfrac{t^2}{2} + \dfrac{t^4}{24}\right)$

21.) $e^{at}(b \cos \omega t + \omega^{-1}(ab + c) \sin \omega t)$

22.) $f(t) = -1$ if $1 < t < 3$

23.) $f(t) = 0$ if $t < 1$ and $t-1$ if $t > 1$

24.) $f(t) = t - 1, 2t - 3, 6 - t$ when $1 < t < 2$, $2 < t < 3$, $3 < t < 6$, and 0 otherwise

25.) $f(t) = -1$ if $0 < t < a$ and 1 if $t > a$

26.) $u_2(t)[1]$

27.) $u_1(t)[(t-1)^2/2]$

28.) $u_2(t)[e^{2(t - 2)}]$

29.) $-u_1(t)[\pi^{-1}\sin \pi t]$

30.) $u_1(t)[e^{3(t - 1)}]$

31.) $(\sin 2t)/2$ if $0 < t < \pi$ and 0 otherwise

32.) $-e^{\pi-t}$ sin t if $t > \pi$ and 0 otherwise

33.) cos t if $0 < t < \pi$ and 0 otherwise

34.) te^{4t}

35.) t sin t

36.) t^2e^{at}

37.) $e^{-t}(\sin t)/t$

38.) $(\sin \omega t)/t$

39.) $(e^t - 1)/t$

40.) $(2 \sinh at)/t$

41.) $(e^{-bt} - e^{-at})/t$

42.) $2(e^t - \cos t)/t$

43.) $e^{at} * e^{at} = te^{at}$

44.) $e^{at}t^2/2$

45.) $(e^{at} - e^{bt})/(a - b)$

46.) $(1 - \cos \omega t)/\omega^2$

47.) $(\omega t - \sin \omega t)/\omega^3$

48.) $(\sin t - t \cos t)/2$

49.) $(t \sin \omega t)/2\omega$

50.) $(\sin t + t \cos t)/2$

51.) $(\cos at - \cos bt)/(b^2 - a^2)$

52.) $2 + e^t$

53.) $1 + 3 \sinh 3t$

54.) $4 \cos 4t + \sin 4t$

55.) $e^{-t}(\cos t - \sin t)$

56.) $(1 - t)e^{-t}$

57.) $(1 + t^2 - 2t^3)e^{-2t}$

58.) $2e^{2t} + te^t$

59.) $te^{-t}\cos t$

60.) $te^{2t}(\cos t - \sin t)$

61.) $e^t(t \cos 2t + \sin 2t)$

62.) $e^t(\cos t + t \sin t)$

Dirac Delta Function

Suppose that a force of magnitude F acts on an object instantaneously. The goal is to describe mathematically a force that imparts an impulse of unit magnitude to an object at time t. (You can always multiply by a magnitude.) This is equivalent to being able to describe a metal bar being hit by a hammer.

The unit impulse function or Dirac delta function is defined in the following way.

1.) $\delta_a(t) = 0, \quad t \neq a$

2.) $\int_{-\infty}^{\infty} \delta_a(t)\, dt = 1$

The Laplace transform $L(\delta_a(t)) = e^{-sa}$

Find the answer to the following.

1.) $y' - 2y = \delta_2(t), \quad y(0) = 1$

2.) $y' + 4y = 3\delta_1(t), \quad y(0) = 0$

3.) $y'' - 4y = \delta_3(t), \quad y(0) = 0, \; y'(0) = 1$

4.) $y'' + 16y = 4\cos 3t + \delta_a(t)$, where $a = \frac{\pi}{3}$ and $y(0) = 0, \; y'(0) = 0$

Answers:

1.) $y = e^{2t} + u_2(t)[e^{2(t-2)}]$

2.) $y = \frac{1}{3}(e^{5t} - e^{-t}) + u_3(t)[e^{5(t-3)}]$

3.) $y = \frac{1}{2}[\sinh 2t + u_3(t)[\sinh 2(t-3)]$

4.) $y = \frac{4}{7}(\cos 3t - \cos 4t) + \frac{1}{4}\, u_a(t)[\sin 4(t-a)]$ where $a = \frac{\pi}{3}$

Convolution

When you have a transform of the form H(s) = F(s) G(s) or what appears to be a product in the world of Laplace, one option is to use convolution.

If $H(s) = F(s)\,G(s)$ Then $h(t) = f(t) * g(t)$ where the star represents convolution.

$$f(t) * g(t) = \int_0^t f(t-\tau)g(\tau)d\tau$$

Sometimes this is useful in finding the inverse transformation. You have several in the exercises.

Periodic Functions

The Laplace transform of a piecewise continuous periodic function f(t) with period p is

$$L(f) = \frac{1}{1 - e^{-ps}} \int_0^p e^{-st} f(t)\,dt$$

This is useful if the function pulse form repeats. There are problems on the exercise sheet.